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TRIANGULAR LIBRATION POINTS AND THEIR STABILITY IN THE RESTRICTED, CIRCULAR, PLANE THREE-BODY PROBLEM*

I.D. SAKHOKIYA and R.S. SULIKASHVILI

Tbilisi

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The effect of the non-sphericity of a body of specific form on the type of relative equilibrium of a passively gravitating body and on the conditions for their stability in a restricted, circular plane three-body problem is studied (it is assumed that the mass of one of the bodies is negligibly small compared with the mass of the other two bodies and has no effect on their motion).

Rectilinear libration points of a passively gravitating point in the field of attraction of a sphere with a spherical distribution of masses and a homogeneous rod, were studied in /1/, where it was assumed that the centres of the rod and the sphere were at a constant distance from each other, that the sphere and the rod rotated with constant angular velocity about the common centre of mass and that the rod was collinear with the radius vector.

In this paper we investigate the triangular libration points of a passively gravitating point in the field of attraction of a sphere and a rod, under the same assumptions concerning the motions of the sphere and rod. It is shown that taking into account the size of the rod leads to displacement of the classical triangular libration points corresponding to a rod of zero length, towards the rod in both directions (i.e. towards the straight line connecting the centres of mass of the sphere and rod, as well as towards the straight line orthogonal to it and passing through the centre of the rod).

1. Let us consider the motion of material points of negligible mass attracted by the sphere and the rod executing stationary motion in which the distance between their centres of mass is constant. The angular velocity of rotation of the sphere and rod about their common centre of mass is also constant and the rod is situated along the radius vector.

Such motion will always exist, provided that /1/

$$\omega^2 = f(M+m)/[\rho(\rho^2 - l^2)] \tag{1.1}$$

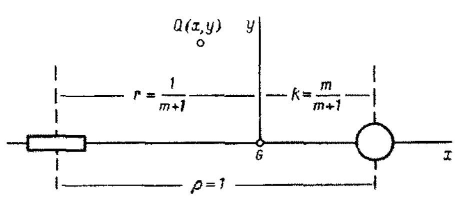
and is stable if

$$3M\rho^4 - 3(4M+m)\rho^2 l^2 + (M+m)l^4 > 0 \tag{1.2}$$

Here f is the universal gravitational constant, M and m are the masses of the sphere and rod respectively, ρ is the distance between the mass centres, $2l$ is rod length and ω is the angular velocity.

It is clear that the inequality (1.2) holds a priori when $l \ll \rho$, i.e. when the rod dimensions are small compared with the distance between the centres of mass of the sphere and rod.

Let the measurement units be chosen so that $f=1$, $M=1$, $\rho=1$. Then, according to (1.1) $\omega^2 = (1+m)/(1-l^2)$ (here, what was said above implies that $l \ll 1$).



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Let us introduce a Gxy coordinate system rotating with angular velocity ω with the origin at the centre of mass of the sphere-rod system and the axis Gx directed towards the sphere (see the figure).

The kinetic energy and force function of the passively gravitating point $Q(x, y)$ of mass ε, have the form

$$2T = \varepsilon[(x^{\cdot} - \omega y)^2 + (y^{\cdot} + \omega x)^2] \quad (x^{\cdot} = dx/dt)$$

$$U = \varepsilon\left[\frac{1}{\sqrt{(x-R)^2+y^2}} + \frac{m}{2l}\int_{-l}^{l}\frac{d\tau}{\sqrt{\zeta^2(\tau)+y^2}}\right], \quad \zeta(\tau) = x + r - \tau,$$

It follows that the equations of motion of the point Q do not depend on ε and have the form

$$x^{\cdot\cdot} - 2\omega y^{\cdot} + \partial W/\partial x = 0, \quad y^{\cdot\cdot} + 2\omega x^{\cdot} + \partial W/\partial y = 0 \tag{1.3}$$

$$W = -\frac{1}{2}\omega^2(x^2+y^2) - \frac{1}{\sqrt{(x-R)^2+y^2}} - \frac{m}{2l}\ln\frac{\zeta(l)+\sqrt{\zeta^2(l)+y^2}}{\zeta(-l)+\sqrt{\zeta^2(-l)+y^2}}$$

where W is the changed potential energy.

The equations of relative equilibria

$$\partial W/\partial x = 0, \quad \partial W/\partial y = 0 \tag{1.4}$$

have solutions $x = x_i, y = 0$ $(i = 1, 2, 3, 4)$ investigated earlier in /1/, and rectilinear libration points correspond to these solutions.

2. Let us consider Eqs.(1.4) for $y \neq 0$. Expanding ω and W in series in powers of $l \ll 1$ and retaining terms up to and including the second order of smallness, we obtain

$$\omega^2 = (1+m)(1+l^2) \tag{2.1}$$

$$W = W_2 = -\frac{1}{2}\omega^2(x^2+y^2) - \frac{1}{\xi} - \frac{m}{\eta} - \frac{m}{6}\frac{2(x+r)^2-y^2}{\eta^5}l^2 + o(l^2)$$

$$\xi = \sqrt{(x-R)^2+y^2}, \quad \eta = \sqrt{(x+r)^2+y^2}$$

The equations of relative equilibria, within the stated approximation, will have the form

$$\omega^2 x = \frac{x-R}{\xi^3} + m\frac{x+r}{\eta^3} + \frac{m}{2}\frac{(x+r)[2(x+r)^2-3y^2]}{\eta^7}l^2 \tag{2.2}$$

$$\omega^2 = \frac{1}{\xi^3} + \frac{m}{\eta^3} + \frac{m}{2}\frac{4(x+r)^2-y^2}{\eta^7}l^2$$

When $l = 0$, Eqs.(2.2) have the solutions /2/

$$x_0 = -\tfrac{1}{2}(1-m)/(1+m), \quad y_0 = \pm\tfrac{1}{2}\sqrt{3} \tag{2.3}$$

When $l \neq 0$, we shall seek the solution of (2.2) in the form of series

$$x = x_0 + x_1 l + \tfrac{1}{2}x_2 l^2 + \ldots, \quad y = \pm(y_0 + y_1 l + \tfrac{1}{2}y_2 l^2 + \ldots) \tag{2.4}$$

Substituting (2.4) into (2.2), taking into account (2.3) and equating the coefficients of like powers of l, we obtain

$$x_1 = 0, \quad y_1 = 0; \quad x_2 = -\frac{3+4m}{24}, \quad y_2 = -\frac{19-4m}{24\sqrt{3}}$$

Thus the triangular libration points are given, in the approximation used, by the coordinates

$$x = -\frac{1}{2}\frac{1-m}{1+m} - \frac{3+4m}{24}l^2, \quad y = \pm\left(\frac{\sqrt{3}}{2} - \frac{19-4m}{24\sqrt{3}}l^2\right) \tag{2.5}$$

i.e. they are distributed symmetrically about the Gx axis.

It is clear that $x_2 < 0$, $y_2 < 0$. Consequently, taking into account the rod length will result in the displacement of triangular libration points (as compared with the classical $x = x_0, y = \pm y_0$) towards the rod in both directions (to the left and downwards, see the figure).

3. Let us inspect the stability of the triangular libration points (2.4) obtained here. By virtue of relations (1.3) and (2.1) the first approximation equations have the form

$$x^{\cdot\cdot} - 2\omega y^{\cdot} + ax + cy = 0,\ y^{\cdot\cdot} + 2\omega x^{\cdot} + by + cx = 0 \tag{3.1}$$

$$a \equiv \frac{\partial^2 W}{\partial x^2} = -\frac{3}{4}(1+m) - \frac{26+3m-8m^2}{16} l^2$$

$$b \equiv \frac{\partial^2 W}{\partial y^2} = -\frac{9}{4}(1+m) - \frac{22+29m-8m^2}{16} l^2$$

$$c \equiv \frac{\partial^2 W}{\partial x\, \partial y} = \frac{3\sqrt{3}}{4}(1+m) + \frac{50-31m+8m^2}{16\sqrt{3}} l^2$$

The following relations hold:

$$a + b = -3(1+m) + (3+2m-m^2) l^2 < 0$$

$$ab - c^2 = \frac{27}{4} m + \frac{3}{16}(75m - 18m^2 - 4m^3) l^2 > 0$$

$$a + b + 4\omega^2 = (1+m^2) + (1+m)^2 l^2 > 0$$

Therefore, the triangular libration points are unstable in the time sense, although the degree of instability is even, i.e. gyroscopic stabilization is possible.

If the condition

$$(a+b+4\omega^2)^2 - 4(ab-c^2) = (1+m)^2 - 27m + {}^1/_4 (8 + 249m - 30m^2 - 4m^3) l^2 > 0 \tag{3.2}$$

holds, then the roots of the characteristic equation of the system (3.1) will be purely imaginary and triangular libration points will be stable to a first approximation.

Since the system in question is Hamiltonian and has two degrees of freedom, it follows that we can assert, by analogy with /3/, that when condition (3.2) holds, triangular libration points are Lyapunov-stable for nearly all values of the parameters m and l (since they have this property at $l = 0$ /3/).

We note that for sufficiently small values of m the condition of stability (3.2) is stronger than the analogous conditions of stability of triangular libration points in the classical formulation of the problem.

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